Terminal Control of Boundary Models [1]

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Consider the problem of extremal mapping: to find a fixed point $v^* \in W$, satisfying extreme inclusion

$$v^* \in \text{Argmin}\{\Phi(v^*, w) + \varphi(w) |\ w \in W \subset R^n\}. \tag{1}$$

Here $(v, w) \in W \times W \subset R^n \times R^n$, W is convex closed bounded set. The function $\Phi(v, w)+\varphi(w)$ is convex with respect to w for any $v \in W$. In regular case the extremal mapping (1) always has a fixed point. This statement includes many well-known finite-dimensional (static) problems, such as the convex or equilibrium programming problems, saddle-point problems, n-person games with the Nash equilibrium, multicriteria equilibrium problems, saddle-point games, economic equilibrium models, variational inequalities.

These problems are interpreted as mathematical models for objects from various areas of science and practice. It is assumed that the objects are immersed in the environment for which characteristics (properties) change over time. Changes to the environment is described by means of a controlled dynamical system. Objects and their mathematical models are also changing with the changing environment. In this situation, we consider the transfer an object from one state (initial) to another one (terminal) for a finite period of time.

$$v_0^* \in \text{Argmin}\{\Phi_0(v_0^*, v_0) + \varphi_0(v_0) \mid\ A_0 v_0 \leq a_0\}, \tag{2}$$

$$v_1^* \in \text{Argmin}\{\Phi_1(v_1^*, v_1) + \varphi_1(v_1) \mid\ A_1 v_1 \leq a_1, \tag{3}$$

$$\frac{d}{dt}v(t) = D(t)v(t) + B(t)u(t),\ t_0 \leq t \leq t_1, \tag{4}$$

$$v(t_0) = v_0^*,\ v(t_1) = v_1^*,\ u(t) \in U\}.$$

Here sets $W_i, i = 0, 1$ from (1) are given in the form of functional inequalities as constraints. Controls are taken from the ball of the Hilbert space $U = \{u(\cdot) \in L_2^r[t_0, t_1] \mid \parallel u(\cdot) \parallel^2_{L_2^r} \leq \mathrm{C}^2\}$, $D(t), B(t)$ -$n \times n, n \times r$ are continuous time-dependent matrix functions, A_0, A_1 are fixed matrices , a_0, a_1 are given vectors . If $\Phi_0(v_0^*, v_0) \equiv 0, \Phi_1(v_1^*, v_1) \equiv 0$, then (2),(3) is transformed into a convex programming problem . These models (2)-(4) allow us to describe transfer an object from one state to another one for a finite period of time.

Dynamical system (2)-(4) is considered in Hilbert space. It means that up to a set of measure zero, values of the function-control $u(\cdot)$ belong to the set $U \subseteq L_2^r[t_0, t_1]$. In the case where the control run through the entire set $u(\cdot) \in U$, differential system (4) generate a trajectories $v(t), t \in [t_0, t_1]$, which are absolutely continuous functions. Left $v(t_0) = v_0$ and right $v(t_1) = v_1$ ends trajectories describe the set of initial and terminal conditions . In the linear case , these sets are subspaces of R^n, which, in particular, may coincide with their spaces.

System (2)-(4) in the report, is treated as a convex-concave programming problem which is formulated on the direct product of the spaces $R^n \times L_2^n[t_0, t_1] \times L_2^r[t_0, t_1] \times R^n$. In this situation

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it is naturally to introduce two linearized Lagrangian (primal and dual), which have the form [1]

$$\mathcal{L}(v_0^*, v_1^*, p_0, p_1, \psi(\cdot); v_0, v_1, v(\cdot), u(\cdot)) = \langle \nabla_2 \Phi_0(v_0^*, v_0^*) + \nabla \varphi_0(v_0^*), v_0 \rangle$$
$$+\langle p_0, A_0 v_0 - a_0 \rangle + \langle \nabla_2 \Phi_1(v_1^*, v_1^*) + \nabla \varphi_1(v_1^*), v_1 \rangle + \langle p_1, A_1 v_1 - a_1 \rangle$$
$$+ \int_{t_0}^{t_1} \langle \psi(t), D(t)v(t) + B(t)u(t) - \frac{d}{dt} v(t) \rangle dt, \qquad (5)$$

and

$$\mathcal{L}^T(v_0^*, v_1^*, p_0, p_1, \psi(\cdot); v_0, v_1, v(\cdot), u(\cdot)) =$$
$$\langle \nabla_2 \Phi_0(v_0^*, v_0^*) + \nabla \varphi_0(v_0^*) + A^T p_0 + \psi_0, v_0 \rangle + \langle -a_0, p_0 \rangle$$
$$+\langle \nabla_2 \Phi_1(v_1^*, v_1^*) + \nabla \varphi_1(v_1^*) + A^T p_1 - \psi_1, v_1 \rangle + \langle -a_1, p_1 \rangle$$
$$+ \int_{t_0}^{t_1} \langle D^T(t)\psi(t) + \frac{d}{dt}\psi(t), v(t) \rangle + \int_{t_0}^{t_1} \langle B^T(t)\psi(t), u(t) \rangle dt \qquad (6)$$

for all $p_0, p_1 \in R_+^m$, $\psi(\cdot) \in \Psi_2^n[t_0, t_1]$, $(v_0, v_1) \in R^n \times R^n$, $(v(\cdot), u(\cdot)) \in AC^n[t_0, t_1] \times U$, where $AC^n[t_0, t_1]$ is the linear manifold of absolutely continuous functions from $L_2^n[t_0, t_1]$, $\Psi_2^n[t_0, t_1]$ is the class of absolutely continuous functions from $(L_2^n[t_0, t_1])^T$. The linear manifold $AC^n[t_0, t_1]$ is dense everywhere in $L_2^n[t_0, t_1]$, i.a. the closure of $AC^n[t_0, t_1]$ in norm $L_2^n[t_0, t_1]$ coincides with $L_2^n[t_0, t_1]$. Here v_0^*, v_1^* are parameters of the Lagrangians.

Both Lagrangians have the same saddle points $(p_0^*, p_1^*, \psi^*(\cdot); v_0^*, v_1^*, v^*(\cdot), u^*(\cdot))$. Components of them form primal $(v_0^*, v_1^*, v^*(\cdot), u^*(\cdot))$ and dual $(p_0^*, p_1^*, \psi^*(\cdot))$ solutions for problem (2)-(4).

The report presents the saddle iterative method for calculating the saddle point for both Lagrange functions. It is proved convergence of method to the saddle point for all components of the primal and dual solutions of initial problem (optimal) control [1],[2]. More precisely, the weak convergence is in respect controls, strong convergence is in the phase and dual trajectories, as well as for terminal saddle variables for boundary value problems corresponding to the ends of the time interval.

REFERENCES

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